Sphere packing, lattice packing, and related problems

Abhinav Kumar

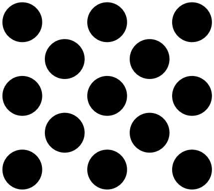
Stony Brook

April 25, 2018

Sphere packings

Definition

A sphere packing in $\mathbb{R}^n$ is a collection of spheres/balls of equal size which do not overlap (except for touching). The density of a sphere packing is the volume fraction of space occupied by the balls.



Sphere packing problem

Problem: Find a/the densest sphere packing(s) in $\mathbb{R}^n$.

Sphere packing problem

Problem: Find a/the densest sphere packing(s) in $\mathbb{R}^n$.

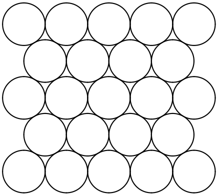
In dimension 1, we can achieve density 1 by laying intervals end to end.

Sphere packing problem

Problem: Find a/the densest sphere packing(s) in $\mathbb{R}^n$.

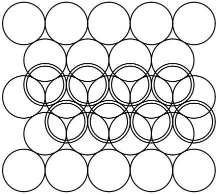
In dimension 1, we can achieve density 1 by laying intervals end to end.

In dimension 2, the best possible is by using the hexagonal lattice. [Fejes Tóth 1940]



Sphere packing problem II

In dimension 3, the best possible way is to stack layers of the solution in 2 dimensions. This is Kepler's conjecture, now a theorem of Hales and collaborators.



There are infinitely (in fact, uncountably) many ways of doing this! These are the Barlow packings.

Face centered cubic packing

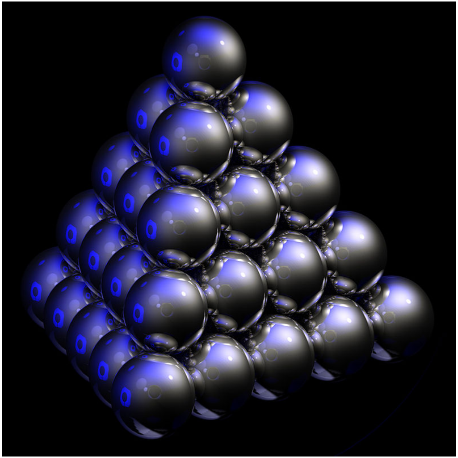


Image: Greg A L (Wikipedia), CC BY-SA 3.0 license

Higher dimensions

In some higher dimensions, we have guesses for the densest sphere packings.

Most of them arise from lattices.

Higher dimensions

In some higher dimensions, we have guesses for the densest sphere packings.

Most of them arise from lattices.

But (until very recently!) no proofs.

Higher dimensions

In some higher dimensions, we have guesses for the densest sphere packings.

Most of them arise from lattices.

But (until very recently!) no proofs.

In very high dimensions (say ≥ 1000) densest packings are likely to be close to disordered.

Large dimensions, upper bounds

For general n, we only have upper and lower bounds on the density of packings in $\mathbb{R}^n$.

Large dimensions, upper bounds

For general n, we only have upper and lower bounds on the density of packings in $\mathbb{R}^n$.

The best systematic upper bound is due to Kabatiansky-Levenshtein, and it looks like $2^{-0.599n}$.

Large dimensions, upper bounds

For general n, we only have upper and lower bounds on the density of packings in $\mathbb{R}^n$.

The best systematic upper bound is due to Kabatiansky-Levenshtein, and it looks like $2^{-0.599n}$.

Cohn-Elkies linear programming bounds were shown to be as good (by Cohn-Zhao).

Problem

Large dimensions, upper bounds

For general n, we only have upper and lower bounds on the density of packings in $\mathbb{R}^n$.

The best systematic upper bound is due to Kabatiansky-Levenshtein, and it looks like $2^{-0.599n}$.

Cohn-Elkies linear programming bounds were shown to be as good (by Cohn-Zhao).

Problem

- *Improve the exponent of the upper bound (if possible).*

Large dimensions, upper bounds

For general n, we only have upper and lower bounds on the density of packings in $\mathbb{R}^n$.

The best systematic upper bound is due to Kabatiansky-Levenshtein, and it looks like $2^{-0.599n}$.

Cohn-Elkies linear programming bounds were shown to be as good (by Cohn-Zhao).

Problem

- *Improve the exponent of the upper bound (if possible).*
- *What is the true exponent of the Cohn-Elkies LP bound?*

Large dimensions, upper bounds

For general n, we only have upper and lower bounds on the density of packings in $\mathbb{R}^n$.

The best systematic upper bound is due to Kabatiansky-Levenshtein, and it looks like $2^{-0.599n}$.

Cohn-Elkies linear programming bounds were shown to be as good (by Cohn-Zhao).

Problem

- *Improve the exponent of the upper bound (if possible).*
- *What is the true exponent of the Cohn-Elkies LP bound?*
- *Can it be systematically improved by using an SDP bound?*

Lower bounds

On the other hand, there is an easy lower bound due to Minkowski of 2^{-n}: take any saturated packing, where you cannot add any more spheres. Doubling radius must cover space.

Lower bounds

On the other hand, there is an easy lower bound due to Minkowski of 2^{-n}: take any saturated packing, where you cannot add any more spheres. Doubling radius must cover space.

Minkowski-Hlawka showed you could get lattice packings with at least this density.

Lower bounds

On the other hand, there is an easy lower bound due to Minkowski of 2^{-n}: take any saturated packing, where you cannot add any more spheres. Doubling radius must cover space.

Minkowski-Hlawka showed you could get lattice packings with at least this density.

The best lower bounds to date are of the form $C \cdot n \cdot 2^{-n}$ for general n. Venkatesh recently showed you could get $C \cdot n \cdot \log \log n \cdot 2^{-n}$ for some very special n.

Problem

Can the lower bound be improved asymptotically?

Lower bounds

On the other hand, there is an easy lower bound due to Minkowski of 2^{-n}: take any saturated packing, where you cannot add any more spheres. Doubling radius must cover space.

Minkowski-Hlawka showed you could get lattice packings with at least this density.

The best lower bounds to date are of the form $C \cdot n \cdot 2^{-n}$ for general n. Venkatesh recently showed you could get $C \cdot n \cdot \log \log n \cdot 2^{-n}$ for some very special n.

Problem

Can the lower bound be improved asymptotically?

Conjecture (Zassenhaus)

In every dimension, the maximal density is attained by a periodic packing.

Lattices

Definition

A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of rank n, i.e. generated by n linearly independent vectors of $\mathbb{R}^n$.

Lattices

Definition

A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of rank n, i.e. generated by n linearly independent vectors of $\mathbb{R}^n$.

Examples

Lattices

Definition

A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of rank n, i.e. generated by n linearly independent vectors of $\mathbb{R}^n$.

Examples

- Integer lattice $\mathbb{Z}^n$.

Lattices

Definition

A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of rank n, i.e. generated by n linearly independent vectors of $\mathbb{R}^n$.

Examples

- Integer lattice $\mathbb{Z}^n$.
- Checkerboard lattice $D_n = \{x \in \mathbb{Z}^n : \sum x_i \text{ even } \}$

Lattices

Definition

A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of rank n, i.e. generated by n linearly independent vectors of $\mathbb{R}^n$.

Examples

- Integer lattice $\mathbb{Z}^n$.
- Checkerboard lattice $D_n = \{x \in \mathbb{Z}^n : \sum x_i \text{ even } \}$
- Simplex lattice $A_n = \{x \in \mathbb{Z}^{n+1} : \sum x_i = 0\}$

Lattices

Definition

A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of rank n, i.e. generated by n linearly independent vectors of $\mathbb{R}^n$.

Examples

- Integer lattice $\mathbb{Z}^n$.
- Checkerboard lattice $D_n = \{x \in \mathbb{Z}^n : \sum x_i \text{ even } \}$
- Simplex lattice $A_n = \{x \in \mathbb{Z}^{n+1} : \sum x_i = 0\}$
- Special root lattices E_6, E_7, E_8.

Lattices

Definition

A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of rank n, i.e. generated by n linearly independent vectors of $\mathbb{R}^n$.

Examples

- Integer lattice $\mathbb{Z}^n$.
- Checkerboard lattice $D_n = \{x \in \mathbb{Z}^n : \sum x_i \text{ even } \}$
- Simplex lattice $A_n = \{x \in \mathbb{Z}^{n+1} : \sum x_i = 0\}$
- Special root lattices E_6, E_7, E_8.
  - E_8 generated by D_8 and all-halves vector.

Lattices

Definition

A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of rank n, i.e. generated by n linearly independent vectors of $\mathbb{R}^n$.

Examples

- Integer lattice $\mathbb{Z}^n$.
- Checkerboard lattice $D_n = \{x \in \mathbb{Z}^n : \sum x_i \text{ even } \}$
- Simplex lattice $A_n = \{x \in \mathbb{Z}^{n+1} : \sum x_i = 0\}$
- Special root lattices E_6, E_7, E_8.
  - E_8 generated by D_8 and all-halves vector.
  - E_7 orthogonal complement of a root (or A_1) in E_8.

Lattices

Definition

A lattice Λ in $\mathbb{R}^n$ is a discrete subgroup of rank n, i.e. generated by n linearly independent vectors of $\mathbb{R}^n$.

Examples

- Integer lattice $\mathbb{Z}^n$.
- Checkerboard lattice $D_n = \{x \in \mathbb{Z}^n : \sum x_i \text{ even } \}$
- Simplex lattice $A_n = \{x \in \mathbb{Z}^{n+1} : \sum x_i = 0\}$
- Special root lattices E_6, E_7, E_8.
  - E_8 generated by D_8 and all-halves vector.
  - E_7 orthogonal complement of a root (or A_1) in E_8.
  - E_6 orthogonal complement of an A_2 in E_8.

Projection of E8 root system

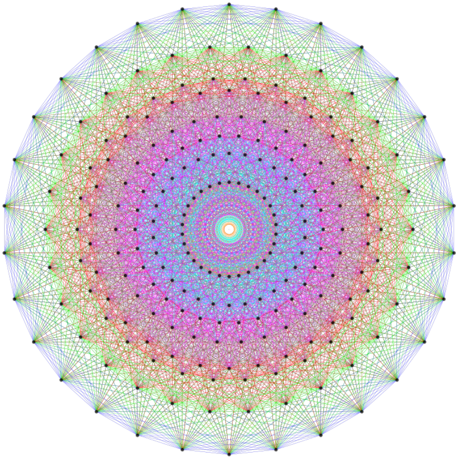


Image: Jgmoxness (Wikipedia), CC BY-SA 3.0 license

Leech lattice

In dimension 24, there is also the remarkable Leech lattice. It is the unique even unimodular lattice in that dimension without any roots.

Leech lattice

In dimension 24, there is also the remarkable Leech lattice. It is the unique even unimodular lattice in that dimension without any roots.

There are many neat constructions of it (for instance, Conway-Sloane give twenty-three constructions). The usual one involves the extended Golay code.

Leech lattice

In dimension 24, there is also the remarkable Leech lattice. It is the unique even unimodular lattice in that dimension without any roots.

There are many neat constructions of it (for instance, Conway-Sloane give twenty-three constructions). The usual one involves the extended Golay code.

My favorite: The lattice $\mathrm{II}_{25,1}$ is generated in $\mathbb{R}^{25,1}$ (which has the quadratic form $x_1^2 + \cdots + x_{25}^2 - x_{26}^2$) by vectors in $\mathbb{Z}^{26}$ or $(\mathbb{Z} + 1/2)^{26}$ with even coordinate sum.

Leech lattice

In dimension 24, there is also the remarkable Leech lattice. It is the unique even unimodular lattice in that dimension without any roots.

There are many neat constructions of it (for instance, Conway-Sloane give twenty-three constructions). The usual one involves the extended Golay code.

My favorite: The lattice $\mathrm{II}_{25,1}$ is generated in $\mathbb{R}^{25,1}$ (which has the quadratic form $x_1^2 + \cdots + x_{25}^2 - x_{26}^2$) by vectors in $\mathbb{Z}^{26}$ or $(\mathbb{Z} + 1/2)^{26}$ with even coordinate sum.

The Weyl vector $w = (0, 1, 2, \ldots, 24, 70)$ has norm 0, since $1^2 + \cdots + 24^2 = 70^2$ (!)

Leech lattice

In dimension 24, there is also the remarkable Leech lattice. It is the unique even unimodular lattice in that dimension without any roots.

There are many neat constructions of it (for instance, Conway-Sloane give twenty-three constructions). The usual one involves the extended Golay code.

My favorite: The lattice $\mathrm{II}_{25,1}$ is generated in $\mathbb{R}^{25,1}$ (which has the quadratic form $x_1^2 + \cdots + x_{25}^2 - x_{26}^2$) by vectors in $\mathbb{Z}^{26}$ or $(\mathbb{Z} + 1/2)^{26}$ with even coordinate sum.

The Weyl vector $w = (0, 1, 2, \ldots, 24, 70)$ has norm 0, since $1^2 + \cdots + 24^2 = 70^2$ (!)

The Leech lattice is $w^\perp / \mathbb{Z}w$ with the induced quadratic form.

Lattice packing

Associated sphere packing: if $m(\Lambda)$ is the length of a smallest non-zero vector of Λ, then we can put balls of radius $m(\Lambda)/2$ around each point of Λ so that they don't overlap.

Lattice packing

Associated sphere packing: if $m(\Lambda)$ is the length of a smallest non-zero vector of Λ, then we can put balls of radius $m(\Lambda)/2$ around each point of Λ so that they don't overlap.

The packing problem for lattices asks for the densest lattice(s) in $\mathbb{R}^n$ for every n. This is equivalent to the determination of the Hermite constant γ_n, which arises in the geometry of numbers. The known answers are:

n	1	2	3	4	5	6	7	8	24
Λ	A_1	A_2	A_3	D_4	D_5	E_6	E_7	E_8	Leech
due to		Lagrange	Gauss	Korkine-Zolotareff		Blichfeldt			Cohn-Kumar

Lattices vs. non-lattices

The best packings that we seem to be able to construct in high dimensions are lattices.

Lattices vs. non-lattices

The best packings that we seem to be able to construct in high dimensions are lattices.

Conjecture (folklore)

For high dimensions, the densest packings should be non-lattice.

Lattices vs. non-lattices

The best packings that we seem to be able to construct in high dimensions are lattices.

Conjecture (folklore)

For high dimensions, the densest packings should be non-lattice.

For instance, the best known packing in dimension 10 is the Best packing, which has 40 translates of a lattice.

Lattices vs. non-lattices

The best packings that we seem to be able to construct in high dimensions are lattices.

Conjecture (folklore)

For high dimensions, the densest packings should be non-lattice.

For instance, the best known packing in dimension 10 is the Best packing, which has 40 translates of a lattice.

In dimension 9, we have the fluid diamond packings.

Lattices vs. non-lattices

The best packings that we seem to be able to construct in high dimensions are lattices.

Conjecture (folklore)

For high dimensions, the densest packings should be non-lattice.

For instance, the best known packing in dimension 10 is the Best packing, which has 40 translates of a lattice.

In dimension 9, we have the fluid diamond packings.

But we don't know a single dimension when this conjecture is proved.

Lattices, quadratic forms

Lattices and quadratic forms are two ways of viewing the same object.

Lattices, quadratic forms

Lattices and quadratic forms are two ways of viewing the same object.

- Euclidean lattice up to isometry, with a basis

Lattices, quadratic forms

Lattices and quadratic forms are two ways of viewing the same object.

- Euclidean lattice up to isometry, with a basis
- Positive definite matrix

Lattices, quadratic forms

Lattices and quadratic forms are two ways of viewing the same object.

- Euclidean lattice up to isometry, with a basis
- Positive definite matrix
- Quadratic form

Lattices, quadratic forms

Lattices and quadratic forms are two ways of viewing the same object.

- Euclidean lattice up to isometry, with a basis
- Positive definite matrix
- Quadratic form

Lattices, quadratic forms

Lattices and quadratic forms are two ways of viewing the same object.

- Euclidean lattice up to isometry, with a basis
- Positive definite matrix
- Quadratic form

So lattices up to isometry are the same as quadratic forms up to invertible integer linear transformation of variables.

$$O(n)\backslash GL(n,\mathbb{R})/GL(n,\mathbb{Z}) \cong GL(n,\mathbb{Z})\backslash Sym^+(n,\mathbb{R})$$

Hermite constant

The question of finding the densest lattice is equivalent to finding the Hermite constant, in any dimension.

For a positive definite quadratic form Q, let $\min(Q)$ be the smallest nonzero value attained by Q when the variables are integers.

Hermite constant

The question of finding the densest lattice is equivalent to finding the Hermite constant, in any dimension.

For a positive definite quadratic form Q, let $\min(Q)$ be the smallest nonzero value attained by Q when the variables are integers.

Definition

The Hermite constant γ_n is the maximum of $\min(Q)$ as Q ranges over pos. def. quadratic forms of determinant 1 and dimension n.

Hermite constant

The question of finding the densest lattice is equivalent to finding the Hermite constant, in any dimension.

For a positive definite quadratic form Q, let $\min(Q)$ be the smallest nonzero value attained by Q when the variables are integers.

Definition

The Hermite constant γ_n is the maximum of $\min(Q)$ as Q ranges over pos. def. quadratic forms of determinant 1 and dimension n.

Voronoi's theorem

Theorem (Voronoi)

A lattice is a local maximum for density iff it is perfect and eutactic.

Voronoi's theorem

Theorem (Voronoi)

A lattice is a local maximum for density iff it is perfect and eutactic.

Let $S(\Lambda) = \{u_1, \ldots, u_N\}$ be the set of minimal vectors of Λ, i.e. those of smallest positive norm in Λ.

Voronoi's theorem

Theorem (Voronoi)

A lattice is a local maximum for density iff it is perfect and eutactic.

Let $S(\Lambda) = \{u_1, \ldots, u_N\}$ be the set of minimal vectors of Λ, i.e. those of smallest positive norm in Λ.

Definition

We say Λ is perfect if the N rank one $n \times n$ matrices $u_i u_i^T$ span the space of symmetric matrices (which has dimension $n(n+1)/2$).

Voronoi's theorem

Theorem (Voronoi)

A lattice is a local maximum for density iff it is perfect and eutactic.

Let $S(\Lambda) = \{u_1, \ldots, u_N\}$ be the set of minimal vectors of Λ, i.e. those of smallest positive norm in Λ.

Definition

We say Λ is perfect if the N rank one $n \times n$ matrices $u_i u_i^T$ span the space of symmetric matrices (which has dimension $n(n+1)/2$).

Definition

We say Λ is eutactic if the identity matrix lies in the positive cone spanned by these rank one matrices.

Perfect forms

One can try to enumerate perfect forms in low dimensions, using an algorithm of Voronoi. Then we can compute which ones are also eutactic, which gives us the set of local optima.

Perfect forms

One can try to enumerate perfect forms in low dimensions, using an algorithm of Voronoi. Then we can compute which ones are also eutactic, which gives us the set of local optima.

n	1	2	3	4	5	6	7	8	9
# Perfect forms	1	1	1	2	3	7	33	10916	> 500000
# Local optima	1	1	1	2	3	6	30	2408	??

The enumeration of 8-dimensional perfect forms was completed by Schuermann, Sikirić, and Vallentin in 2009.

Problem

Determine the densest lattices in dimensions 9 *and* 10 *and prove the folklore conjecture that their density is exceeded by non-lattice packings.*

Extremal even unimodular lattices I

One nice class of lattices is that of the even unimodular ones.
These only exist in dimensions that are multiples of 8.

Extremal even unimodular lattices I

One nice class of lattices is that of the even unimodular ones. These only exist in dimensions that are multiples of 8.

The theta function $\sum_{x \in \Lambda} q^{\langle x,x \rangle/2}$ is a modular form $1 + a_1 q + a_2 q^2 + \dots$.

Extremal even unimodular lattices I

One nice class of lattices is that of the even unimodular ones. These only exist in dimensions that are multiples of 8.

The theta function $\sum_{x \in \Lambda} q^{\langle x,x \rangle/2}$ is a modular form $1 + a_1 q + a_2 q^2 + \dots$.

For the lattice to be a good packing, want as many of $a_1, \dots, a_r$ to vanish as possible.

Let $n = 24m + 8k$ with $k \in \{0, 1, 2\}$. Then dimension of space of modular forms gives that $a_1, \dots, a_{2m+2}$ cannot all vanish.

Definition

The (even unimodular) lattice is extremal if $a_1, \dots, a_{2m+1}$ are all 0.

Extremal even unimodular lattices I

One nice class of lattices is that of the even unimodular ones. These only exist in dimensions that are multiples of 8.

The theta function $\sum_{x \in \Lambda} q^{\langle x,x \rangle/2}$ is a modular form $1 + a_1 q + a_2 q^2 + \dots$.

For the lattice to be a good packing, want as many of $a_1, \dots, a_r$ to vanish as possible.

Let $n = 24m + 8k$ with $k \in \{0, 1, 2\}$. Then dimension of space of modular forms gives that $a_1, \dots, a_{2m+2}$ cannot all vanish.

Definition

The (even unimodular) lattice is extremal if $a_1, \dots, a_{2m+1}$ are all 0.

Extremal lattices cannot exist for n larger than ≈ 41000 (the value of a_{2m+2} becomes negative.

Extremal even unimodular lattices II

Till recently, extremal even unimodular lattices were known in dimensions all multiples of 8 through 80, except for dimension 72.

Extremal even unimodular lattices II

Till recently, extremal even unimodular lattices were known in dimensions all multiples of 8 through 80, except for dimension 72.

In 2012, Nebe constructed an extremal even unimodular lattice in dimension 72. Proof involves enumeration of vectors of norm 8, uses symmetry group.

Extremal even unimodular lattices II

Till recently, extremal even unimodular lattices were known in dimensions all multiples of 8 through 80, except for dimension 72.

In 2012, Nebe constructed an extremal even unimodular lattice in dimension 72. Proof involves enumeration of vectors of norm 8, uses symmetry group.

Existence of these in higher dimensions is still open.

Extremal even unimodular lattices II

Till recently, extremal even unimodular lattices were known in dimensions all multiples of 8 through 80, except for dimension 72.

In 2012, Nebe constructed an extremal even unimodular lattice in dimension 72. Proof involves enumeration of vectors of norm 8, uses symmetry group.

Existence of these in higher dimensions is still open.

For the total number of even unimodular lattices in a given dimension, one can use the Siegel mass formula to give a lower bound which grows very rapidly. In fact, the number of extremal ones also seems to initially grow quite rapidly.

One in $\mathbb{R}^8$, two in $\mathbb{R}^{16}$, one in $\mathbb{R}^{24}$, at least 10^7 in $\mathbb{R}^{32}$, at least 10^{51} in $\mathbb{R}^{40}$.

Kissing problem I

The kissing number problem asks for the smallest number of unit spheres which can touch a central unit sphere, without overlapping.

In $\mathbb{R}^3$, this is called the Gregory-Newton problem. Newton believed the answer was 12, whereas Gregory thought you could fit a thirteenth sphere.

Kissing problem I

The kissing number problem asks for the smallest number of unit spheres which can touch a central unit sphere, without overlapping.

In $\mathbb{R}^3$, this is called the Gregory-Newton problem. Newton believed the answer was 12, whereas Gregory thought you could fit a thirteenth sphere.

Newton was correct. Proof by Schütte van der Waerden around 1950.

Kissing problem I

The kissing number problem asks for the smallest number of unit spheres which can touch a central unit sphere, without overlapping.

In $\mathbb{R}^3$, this is called the Gregory-Newton problem. Newton believed the answer was 12, whereas Gregory thought you could fit a thirteenth sphere.

Newton was correct. Proof by Schütte van der Waerden around 1950.

Leech gave a short proof, which was also used for the first chapter of "Proofs from the Book", but it omitted so many details it was later scrapped.

n	1	2	3	4	5	6	7	8	24
Kissing number	2	6	12	24	?	?	?	240	196560

Kissing problem II

The answers in 8 and 24 dimensions are unique and come from the E_8 and Leech lattices.

Kissing problem II

The answers in 8 and 24 dimensions are unique and come from the E_8 and Leech lattices.

They were proved by Odlyzko-Sloane and Levenshtein using linear programming bounds. Uniqueness by Bannai and Sloane.

Kissing problem II

The answers in 8 and 24 dimensions are unique and come from the E_8 and Leech lattices.

They were proved by Odlyzko-Sloane and Levenshtein using linear programming bounds. Uniqueness by Bannai and Sloane.

Kissing number in $\mathbb{R}^4$: proved by Musin using LP bounds and geometric reasoning (2003).

Kissing problem II

The answers in 8 and 24 dimensions are unique and come from the E_8 and Leech lattices.

They were proved by Odlyzko-Sloane and Levenshtein using linear programming bounds. Uniqueness by Bannai and Sloane.

Kissing number in $\mathbb{R}^4$: proved by Musin using LP bounds and geometric reasoning (2003).

Different proof by Bachoc and Vallentin using semidefinite programming bounds.

Open problems for kissing numbers

- Show that the only 24-point kissing configuration in 4 dimensions is that of D_4.

Open problems for kissing numbers

- Show that the only 24-point kissing configuration in 4 dimensions is that of D_4.
- Improve asymptotic lower bounds on kissing numbers. The best bound currently is the Shannon-Wyner bound which grows like $2^{0.2075n}$ in the dimension.

Open problems for kissing numbers

- Show that the only 24-point kissing configuration in 4 dimensions is that of D_4.
- Improve asymptotic lower bounds on kissing numbers. The best bound currently is the Shannon-Wyner bound which grows like $2^{0.2075n}$ in the dimension.
- Improve asymptotic upper bounds on kissing numbers. The best bound at the moment is Kabatiansky-Levenshtein's bound which grows like $2^{0.401n}$.

Open problems for kissing numbers

- Show that the only 24-point kissing configuration in 4 dimensions is that of D_4.
- Improve asymptotic lower bounds on kissing numbers. The best bound currently is the Shannon-Wyner bound which grows like $2^{0.2075n}$ in the dimension.
- Improve asymptotic upper bounds on kissing numbers. The best bound at the moment is Kabatiansky-Levenshtein's bound which grows like $2^{0.401n}$.

Open problems for kissing numbers

- Show that the only 24-point kissing configuration in 4 dimensions is that of D_4.
- Improve asymptotic lower bounds on kissing numbers. The best bound currently is the Shannon-Wyner bound which grows like $2^{0.2075n}$ in the dimension.
- Improve asymptotic upper bounds on kissing numbers. The best bound at the moment is Kabatiansky-Levenshtein's bound which grows like $2^{0.401n}$.

Note that there was a recent breakthrough by Vladut, showing that the maximum lattice kissing number grows exponentially in the dimension.

Spherical codes

Alternative formulation of kissing problem: maximum number of points on a unit sphere which are separated by angles at least $\pi/3$.

Spherical codes

Alternative formulation of kissing problem: maximum number of points on a unit sphere which are separated by angles at least $\pi/3$.

Can replace this by any angle θ: it becomes the spherical coding problem.

Spherical codes

Alternative formulation of kissing problem: maximum number of points on a unit sphere which are separated by angles at least $\pi/3$.

Can replace this by any angle θ: it becomes the spherical coding problem.

Exact answers are known for very few values of (dimension, angle).

Spherical codes

Alternative formulation of kissing problem: maximum number of points on a unit sphere which are separated by angles at least $\pi/3$.

Can replace this by any angle θ: it becomes the spherical coding problem.

Exact answers are known for very few values of (dimension, angle).

They are usually sharp for the linear programming bound and also spherical designs (Delsarte-Goethals-Seidel, Levenshtein).

The new results in sphere packing

Theorem (Viazovska)

The E_8 lattice packing is the densest sphere packing in $\mathbb{R}^8$.

The new results in sphere packing

Theorem (Viazovska)

The E_8 lattice packing is the densest sphere packing in $\mathbb{R}^8$.

The new results in sphere packing

Theorem (Viazovska)

The E_8 lattice packing is the densest sphere packing in $\mathbb{R}^8$.

Theorem (Cohn-Kumar-Miller-Radchenko-Viazovska)

The Leech lattice packing is the densest sphere packing in $\mathbb{R}^{24}$.

The new results in sphere packing

Theorem (Viazovska)

The E_8 lattice packing is the densest sphere packing in $\mathbb{R}^8$.

Theorem (Cohn-Kumar-Miller-Radchenko-Viazovska)

The Leech lattice packing is the densest sphere packing in $\mathbb{R}^{24}$.

The new results in sphere packing

Theorem (Viazovska)

The E_8 lattice packing is the densest sphere packing in $\mathbb{R}^8$.

Theorem (Cohn-Kumar-Miller-Radchenko-Viazovska)

The Leech lattice packing is the densest sphere packing in $\mathbb{R}^{24}$.

The proof is fairly direct, using just two main ingredients:

1. linear programming bounds for packing

The new results in sphere packing

Theorem (Viazovska)

The E_8 lattice packing is the densest sphere packing in $\mathbb{R}^8$.

Theorem (Cohn-Kumar-Miller-Radchenko-Viazovska)

The Leech lattice packing is the densest sphere packing in $\mathbb{R}^{24}$.

The proof is fairly direct, using just two main ingredients:

1. linear programming bounds for packing
2. the theory of modular forms

Linear programming bounds for sphere packing

Let the Fourier transform of a function f be defined by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x,t \rangle} dx.$$

Linear programming bounds for sphere packing

Let the Fourier transform of a function f be defined by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x, t \rangle} dx.$$

Theorem (Cohn-Elkies)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a Schwartz function with the properties

Linear programming bounds for sphere packing

Let the Fourier transform of a function f be defined by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x,t \rangle} dx.$$

Theorem (Cohn-Elkies)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a Schwartz function with the properties

1. $f(0) = \hat{f}(0) = 1.$

Linear programming bounds for sphere packing

Let the Fourier transform of a function f be defined by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x,t \rangle} dx.$$

Theorem (Cohn-Elkies)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a Schwartz function with the properties

1. $f(0) = \hat{f}(0) = 1$.
2. $f(x) \leq 0$ *for* $|x| \geq r$ *(for some number* $r > 0$*).*

Linear programming bounds for sphere packing

Let the Fourier transform of a function f be defined by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x, t \rangle} dx.$$

Theorem (Cohn-Elkies)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a Schwartz function with the properties

1. $f(0) = \hat{f}(0) = 1$.
2. $f(x) \leq 0$ *for* $|x| \geq r$ *(for some number* $r > 0$*).*
3. $\hat{f}(t) \geq 0$ *for all* t.

Linear programming bounds for sphere packing

Let the Fourier transform of a function f be defined by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x,t \rangle} dx.$$

Theorem (Cohn-Elkies)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a Schwartz function with the properties

1. $f(0) = \hat{f}(0) = 1$.
2. $f(x) \leq 0$ *for* $|x| \geq r$ *(for some number* $r > 0$*).*
3. $\hat{f}(t) \geq 0$ *for all* t.

Linear programming bounds for sphere packing

Let the Fourier transform of a function f be defined by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x,t \rangle} dx.$$

Theorem (Cohn-Elkies)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a Schwartz function with the properties

1. $f(0) = \hat{f}(0) = 1$.
2. $f(x) \leq 0$ *for* $|x| \geq r$ *(for some number* $r > 0$*).*
3. $\hat{f}(t) \geq 0$ *for all* t.

Then the density of any sphere packing in $\mathbb{R}^n$ is bounded above by

$$vol(B_n)(r/2)^n.$$

LP bounds

Why is it a linear programming bound?

LP bounds

Why is it a linear programming bound?

We can rephrase the theorem (by scaling f appropriately) to say: if

1. $\hat{f}(0) = 1$

LP bounds

Why is it a linear programming bound?

We can rephrase the theorem (by scaling f appropriately) to say: if

1. $\hat{f}(0) = 1$
2. $f(x) \leq 0$ for $|x| \geq 1$

LP bounds

Why is it a linear programming bound?

We can rephrase the theorem (by scaling f appropriately) to say: if

1. $\hat{f}(0) = 1$
2. $f(x) \leq 0$ for $|x| \geq 1$
3. $\hat{f}(t) \geq 0$ for all t

LP bounds

Why is it a linear programming bound?

We can rephrase the theorem (by scaling f appropriately) to say: if

1. $\hat{f}(0) = 1$
2. $f(x) \leq 0$ for $|x| \geq 1$
3. $\hat{f}(t) \geq 0$ for all t

LP bounds

Why is it a linear programming bound?

We can rephrase the theorem (by scaling f appropriately) to say: if

1. $\hat{f}(0) = 1$
2. $f(x) \leq 0$ for $|x| \geq 1$
3. $\hat{f}(t) \geq 0$ for all t

then the density is bounded by $2^{-n} vol(B_n) f(0)$.

LP bounds

Why is it a linear programming bound?

We can rephrase the theorem (by scaling f appropriately) to say: if

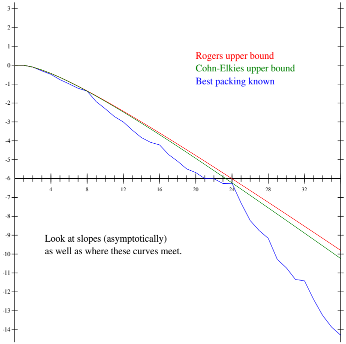
1. $\hat{f}(0) = 1$
2. $f(x) \leq 0$ for $|x| \geq 1$
3. $\hat{f}(t) \geq 0$ for all t

then the density is bounded by $2^{-n} vol(B_n) f(0)$.

Note that the constraints and objective function given are linear in f. Therefore this is a linear (convex) program.

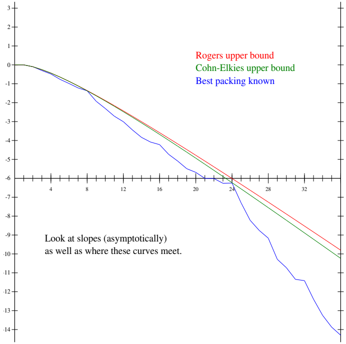
LP bounds with dimension

Here is a plot of $\log(\mathrm{density})$ vs. dimension.



LP bounds with dimension

Here is a plot of $\log(\text{density})$ vs. dimension.

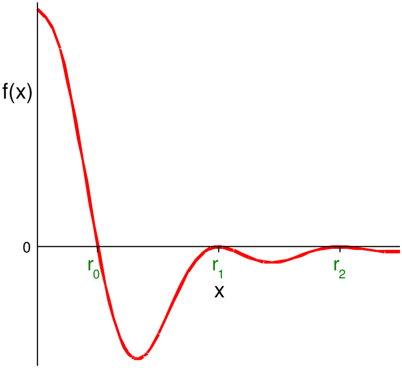


Conjecture (Cohn-Elkies)

There exist "magic" functions f_8 and f_{24} whose corresponding upper bounds match the densities of E_8 and Λ_{24}.

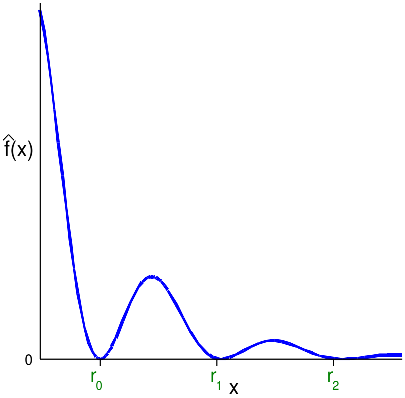
Desired functions

Let Λ be E_8 or the Leech lattice, and $r_0, r_1, \dots$ its nonzero vector lengths (square roots of the even natural numbers, except Leech skips 2). To have a tight upper bound that matches Λ, we need the function f to look like this:



Desired functions

While $\hat{f}$ must look like this:



Impasse

In [Cohn-Kumar] we used a polynomial of degree 803 and 3000 digits of precision to find f and $\hat{f}$ which looked like this with 200 forced double roots, and r very close to 2.

Impasse

In [Cohn-Kumar] we used a polynomial of degree 803 and 3000 digits of precision to find f and $\hat{f}$ which looked like this with 200 forced double roots, and r very close to 2.

Obtained an upper bound of Leech lattice density times $1 + 10^{-30}$. Similar bounds for E_8. Further numerical experimentation by Cohn and Miller.

Impasse

In [Cohn-Kumar] we used a polynomial of degree 803 and 3000 digits of precision to find f and $\hat{f}$ which looked like this with 200 forced double roots, and r very close to 2.

Obtained an upper bound of Leech lattice density times $1 + 10^{-30}$. Similar bounds for E_8. Further numerical experimentation by Cohn and Miller.

But how do we write down exact functions??

Impasse

In [Cohn-Kumar] we used a polynomial of degree 803 and 3000 digits of precision to find f and $\hat{f}$ which looked like this with 200 forced double roots, and r very close to 2.

Obtained an upper bound of Leech lattice density times $1 + 10^{-30}$. Similar bounds for E_8. Further numerical experimentation by Cohn and Miller.

But how do we write down exact functions??

We were stuck for more than a decade.

Viazovska's breakthrough

In March 2016 Maryna Viazovska posted a preprint to the arxiv, solving the sphere packing problem in 8 dimensions.

She found the magic function f_8!

Viazovska's breakthrough

In March 2016 Maryna Viazovska posted a preprint to the arxiv, solving the sphere packing problem in 8 dimensions.

She found the magic function f_8!

Her proof used modular forms.

Viazovska's breakthrough

In March 2016 Maryna Viazovska posted a preprint to the arxiv, solving the sphere packing problem in 8 dimensions.

She found the magic function f_8!

Her proof used modular forms.

Shortly afterward Cohn-Kumar-Miller-Radchenko-Viazovska were able to adapt her ideas to find the magic function f_{24}, solving the 24-dimensional sphere packing problem.

Modular group

A modular form is a function $\phi : \mathcal{H} \to \mathbb{C}$ with a lot of symmetries.

Modular group

A modular form is a function $\phi : \mathcal{H} \to \mathbb{C}$ with a lot of symmetries.

Specifically, let $\mathrm{SL}_2(\mathbb{Z})$ denote all the integer two by two matrices of determinant 1.

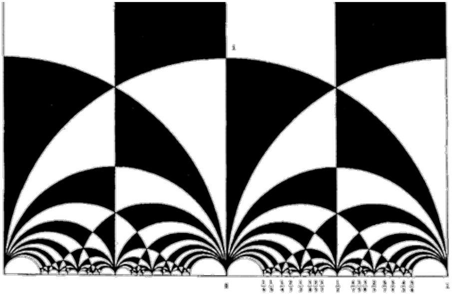


Image from the blog neverendingbooks.org, originally from John Stillwell's article "Modular miracles" in Amer. Math. Monthly.

Fundamental domain

The picture shows Dedekind's famous tesselation of the upper half plane. The union of a black and a white region makes a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$.

Fundamental domain

The picture shows Dedekind's famous tesselation of the upper half plane. The union of a black and a white region makes a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$.

The quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ can be identified with the Riemann sphere $\mathbb{CP}^1$ minus a point.

The preimages of this point are ∞ and the rational numbers; these are the cusps.

Fundamental domain

The picture shows Dedekind's famous tesselation of the upper half plane. The union of a black and a white region makes a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$.

The quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ can be identified with the Riemann sphere $\mathbb{CP}^1$ minus a point.

The preimages of this point are ∞ and the rational numbers; these are the cusps.

We say Γ is a congruence subgroup if it contains all the elements of $\mathrm{SL}_2(\mathbb{Z})$ congruent to the identity modulo N, for some natural number N. Again the quotient is a complex algebraic curve; we can compactify it by adding finitely many cusps.

Modular forms

The first condition for a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ to be a modular form for Γ of weight k is

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

(That is, these are the infinitely many "symmetries" of f.)

Modular forms

The first condition for a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ to be a modular form for Γ of weight k is

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

(That is, these are the infinitely many "symmetries" of f.)

The second condition is a growth condition as we approach a cusp, which we won't describe in detail here.

Examples

How do we find actual examples of modular forms?

Examples

How do we find actual examples of modular forms?

One way is to take simple examples of a "well-behaved" holomorphic function and symmetrize (recalling that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$):

$$G_k(z) = \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(az+b)^k}.$$

Examples

How do we find actual examples of modular forms?

One way is to take simple examples of a "well-behaved" holomorphic function and symmetrize (recalling that $SL_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$):

$$G_k(z) = \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(az+b)^k}.$$

For even $k \geq 4$, the sum converges absolutely and we get a non-zero modular form of weight k. These are called Eisenstein series.

Examples II

Another way is by taking theta functions of integral lattices.

If Λ is a lattice whose inner products $\langle x, y \rangle$ are all integers, then

$$\Theta_\Lambda(z) = \sum_{v \in \Lambda} \exp(\pi i z |v|^2)$$

is a modular form for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Examples II

Another way is by taking theta functions of integral lattices.

If Λ is a lattice whose inner products $\langle x, y \rangle$ are all integers, then

$$\Theta_\Lambda(z) = \sum_{v \in \Lambda} \exp(\pi i z |v|^2)$$

is a modular form for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Fact

The space of modular forms of a given weight for a given congruence subgroup is finite dimensonal.

Sketch of proof

The magic function is constructed using Laplace transforms of suitable modular forms: $f_8 = f_{8,+} + f_{8,-}$, where

$$f_{8,\epsilon} = \sin^2(\pi r^2/2) \int_0^\infty \phi_\epsilon(it) e^{-\pi r^2 t}\, dt$$

for $\epsilon \in \{\pm 1\}$, where ϕ_ϵ is a modular form chosen for a suitable congruence subgroup and suitable weight such that

Sketch of proof

The magic function is constructed using Laplace transforms of suitable modular forms: $f_8 = f_{8,+} + f_{8,-}$, where

$$f_{8,\epsilon} = \sin^2(\pi r^2/2) \int_0^\infty \phi_\epsilon(it) e^{-\pi r^2 t}\, dt$$

for $\epsilon \in \{\pm 1\}$, where ϕ_ϵ is a modular form chosen for a suitable congruence subgroup and suitable weight such that

- The integral has a simple pole at $r^2 = 2$. (Note that $\sin^2(\pi r^2/2)$ has double zeros at $r^2 = 2, 4, 6, 8, \ldots$, so this makes f have the correct "shape".)

Sketch of proof

The magic function is constructed using Laplace transforms of suitable modular forms: $f_8 = f_{8,+} + f_{8,-}$, where

$$f_{8,\epsilon} = \sin^2(\pi r^2/2) \int_0^\infty \phi_\epsilon(it) e^{-\pi r^2 t}\, dt$$

for $\epsilon \in \{\pm 1\}$, where ϕ_ϵ is a modular form chosen for a suitable congruence subgroup and suitable weight such that

- The integral has a simple pole at $r^2 = 2$. (Note that $\sin^2(\pi r^2/2)$ has double zeros at $r^2 = 2, 4, 6, 8, \ldots$, so this makes f have the correct "shape".)
- The symmetry properties of ϕ_ϵ imply that $\widehat{f_{8,\epsilon}} = \epsilon\, f_{8,\epsilon}$, i.e., it is an eigenfunction for the Fourier transform.

Sketch of proof

The magic function is constructed using Laplace transforms of suitable modular forms: $f_8 = f_{8,+} + f_{8,-}$, where

$$f_{8,\epsilon} = \sin^2(\pi r^2/2) \int_0^\infty \phi_\epsilon(it) e^{-\pi r^2 t}\, dt$$

for $\epsilon \in \{\pm 1\}$, where ϕ_ϵ is a modular form chosen for a suitable congruence subgroup and suitable weight such that

- The integral has a simple pole at $r^2 = 2$. (Note that $\sin^2(\pi r^2/2)$ has double zeros at $r^2 = 2, 4, 6, 8, \ldots$, so this makes f have the correct "shape".)
- The symmetry properties of ϕ_ϵ imply that $\widehat{f_{8,\epsilon}} = \epsilon\, f_{8,\epsilon}$, i.e., it is an eigenfunction for the Fourier transform.

Sketch of proof

The magic function is constructed using Laplace transforms of suitable modular forms: $f_8 = f_{8,+} + f_{8,-}$, where

$$f_{8,\epsilon} = \sin^2(\pi r^2/2) \int_0^\infty \phi_\epsilon(it) e^{-\pi r^2 t}\, dt$$

for $\epsilon \in \{\pm 1\}$, where ϕ_ϵ is a modular form chosen for a suitable congruence subgroup and suitable weight such that

- The integral has a simple pole at $r^2 = 2$. (Note that $\sin^2(\pi r^2/2)$ has double zeros at $r^2 = 2, 4, 6, 8, \ldots$, so this makes f have the correct "shape".)
- The symmetry properties of ϕ_ϵ imply that $\widehat{f_{8,\epsilon}} = \epsilon\, f_{8,\epsilon}$, i.e., it is an eigenfunction for the Fourier transform.

One still has to show there are no extra sign changes, which can be accomplished by a computer check of appropriate properties of ϕ.

Future work

There is a broad generalization of sphere packings - the problem of potential energy minimization. Cohn and I conjectured that E_8 and the Leech lattices and the hexagonal lattice A_2 are optimal for a wide range of potentials (and proved analogous statements for many optimal spherical codes).

Critical ingredient is LP bounds for energy (due to Yudin for sphere, Cohn-K for Euclidean space).

Future work

There is a broad generalization of sphere packings - the problem of potential energy minimization. Cohn and I conjectured that E_8 and the Leech lattices and the hexagonal lattice A_2 are optimal for a wide range of potentials (and proved analogous statements for many optimal spherical codes).

Critical ingredient is LP bounds for energy (due to Yudin for sphere, Cohn-K for Euclidean space).

We (the same team of five) have proved this universal optimality for E_8 and Λ_{24} and are in the process of writing it up. For A_2 it remains open!

Future work

There is a broad generalization of sphere packings - the problem of potential energy minimization. Cohn and I conjectured that E_8 and the Leech lattices and the hexagonal lattice A_2 are optimal for a wide range of potentials (and proved analogous statements for many optimal spherical codes).

Critical ingredient is LP bounds for energy (due to Yudin for sphere, Cohn-K for Euclidean space).

We (the same team of five) have proved this universal optimality for E_8 and Λ_{24} and are in the process of writing it up. For A_2 it remains open!

We also have a project to analyze whether these modular-form techniques will lead to asymptotic improvements in upper bounds for sphere packing density.

References

- David de Laat, Frank Vallentin, A Breakthrough in Sphere Packing: The Search for Magic Functions, Nieuw Archief voor Wiskunde (5) 17 (2016), 184-192.
- Henry Cohn, A conceptual breakthrough in sphere packing, Notices Amer. Math. Soc. 64 (2017), no. 2, 102-115.
- Florian Pfender and Günter Ziegler, Kissing numbers, Sphere packings, and some unexpected proofs, Notices. Amer. Math. Soc. 51 (2004), no. 8, 873-883.

References

- David de Laat, Frank Vallentin, A Breakthrough in Sphere Packing: The Search for Magic Functions, Nieuw Archief voor Wiskunde (5) 17 (2016), 184-192.
- Henry Cohn, A conceptual breakthrough in sphere packing, Notices Amer. Math. Soc. 64 (2017), no. 2, 102-115.
- Florian Pfender and Günter Ziegler, Kissing numbers, Sphere packings, and some unexpected proofs, Notices. Amer. Math. Soc. 51 (2004), no. 8, 873-883.

Thank you!