Sphere packing, lattice packing, and related problems

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Sphere packings

Definition

A sphere packing in $\mathbb{R}^n$ is a collection of spheres/balls of equal size which do not overlap (except for touching). The density of a sphere packing is the volume fraction of space occupied by the balls.
Problem: Find a/the densest sphere packing(s) in $\mathbb{R}^n$. 
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In dimension 2, the best possible is by using the hexagonal lattice. [Fejes Tóth 1940]

![Hexagonal lattice diagram]
Sphere packing problem II

In dimension 3, the best possible way is to stack layers of the solution in 2 dimensions. This is Kepler’s conjecture, now a theorem of Hales and collaborators.

There are infinitely (in fact, uncountably) many ways of doing this! These are the Barlow packings.
Face centered cubic packing

Image: Greg A L (Wikipedia), CC BY-SA 3.0 license
Higher dimensions

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In very high dimensions (say $\geq 1000$) densest packings are likely to be close to disordered.
Large dimensions, upper bounds

For general $n$, we only have upper and lower bounds on the density of packings in $\mathbb{R}^n$. 
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- *Improve the exponent of the upper bound (if possible).*
- *What is the true exponent of the Cohn-Elkies LP bound?*
- *Can it be systematically improved by using an SDP bound?*
Lower bounds

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Conjecture (Zassenhaus)

*In every dimension, the maximal density is attained by a periodic packing.*
Lattices

Definition

A lattice $\Lambda$ in $\mathbb{R}^n$ is a discrete subgroup of rank $n$, i.e. generated by $n$ linearly independent vectors of $\mathbb{R}^n$. 

Examples

- Integer lattice $\mathbb{Z}^n$
- Checkerboard lattice $D_n = \{ x \in \mathbb{Z}^n : \sum x_i \text{ even} \}$
- Simplex lattice $A_n = \{ x \in \mathbb{Z}^{n+1} : \sum x_i = 0 \}$
- Special root lattices $E_6, E_7, E_8$. 

- $E_8$ generated by $D_8$ and all-halves vector.
- $E_7$ orthogonal complement of a root (or $A_1$) in $E_8$.
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Projection of E8 root system

Image: Jgmoxness (Wikipedia), CC BY-SA 3.0 license
Leech lattice

In dimension 24, there is also the remarkable Leech lattice. It is the unique even unimodular lattice in that dimension without any roots.
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The Leech lattice is $w^\perp/\mathbb{Z}w$ with the induced quadratic form.
Lattice packing

Associated sphere packing: if $m(\Lambda)$ is the length of a smallest non-zero vector of $\Lambda$, then we can put balls of radius $m(\Lambda)/2$ around each point of $\Lambda$ so that they don’t overlap.
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The packing problem for lattices asks for the densest lattice(s) in $\mathbb{R}^n$ for every $n$. This is equivalent to the determination of the Hermite constant $\gamma_n$, which arises in the geometry of numbers. The known answers are:

<table>
<thead>
<tr>
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<th>1</th>
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<tbody>
<tr>
<td>$\Lambda$</td>
<td>$A_1$</td>
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<td>due to</td>
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Lattices vs. non-lattices

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But we don’t know a single dimension when this conjecture is proved.
Lattices, quadratic forms

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So lattices up to isometry are the same as quadratic forms up to invertible integer linear transformation of variables.

\[
O(n) \backslash GL(n, \mathbb{R}) / GL(n, \mathbb{Z}) \cong GL(n, \mathbb{Z}) \backslash Sym^+(n, \mathbb{R})
\]
Hermite constant

The question of finding the densest lattice is equivalent to finding the Hermite constant, in any dimension.

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We say \( \Lambda \) is perfect if the \( N \) rank one \( n \times n \) matrices \( u_i u_i^T \) span the space of symmetric matrices (which has dimension \( n(n + 1)/2 \)).
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Definition

We say $\Lambda$ is eutactic if the identity matrix lies in the positive cone spanned by these rank one matrices.
Perfect forms

One can try to enumerate perfect forms in low dimensions, using an algorithm of Voronoi. Then we can compute which ones are also eutactic, which gives us the set of local optima.
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<tbody>
<tr>
<td># Perfect forms</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>33</td>
<td>10916</td>
<td>&gt; 500000</td>
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<tr>
<td># Local optima</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
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<td>6</td>
<td>30</td>
<td>2408</td>
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The enumeration of 8-dimensional perfect forms was completed by Schuermann, Sikirić, and Vallentin in 2009.

Problem

*Determine the densest lattices in dimensions 9 and 10 and prove the folklore conjecture that their density is exceeded by non-lattice packings.*
Extremal even unimodular lattices

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Let \( n = 24m + 8k \) with \( k \in \{0, 1, 2\} \). Then dimension of space of modular forms gives that \( a_1, \ldots, a_{2m+2} \) cannot all vanish.

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Extremal lattices cannot exist for \( n \) larger than \( \approx 41000 \) (the value of \( a_{2m+2} \) becomes negative.)
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For the total number of even unimodular lattices in a given dimension, one can use the Siegel mass formula to give a lower bound which grows very rapidly. In fact, the number of extremal ones also seems to initially grow quite rapidly.

One in $\mathbb{R}^8$, two in $\mathbb{R}^{16}$, one in $\mathbb{R}^{24}$, at least $10^7$ in $\mathbb{R}^{32}$, at least $10^{51}$ in $\mathbb{R}^{40}$. 
The **kissing number problem** asks for the smallest number of unit spheres which can touch a central unit sphere, without overlapping.

In $\mathbb{R}^3$, this is called the Gregory-Newton problem. Newton believed the answer was 12, whereas Gregory thought you could fit a thirteenth sphere.
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Leech gave a short proof, which was also used for the first chapter of “Proofs from the Book”, but it omitted so many details it was later scrapped.

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Different proof by Bachoc and Vallentin using semidefinite programming bounds.
Open problems for kissing numbers

- Show that the only 24-point kissing configuration in 4 dimensions is that of $D_4$. 

- Improve asymptotic lower bounds on kissing numbers. The best bound currently is the Shannon-Wyner bound which grows like $2^{0.2075n}$ in the dimension.

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Exact answers are known for very few values of (dimension, angle). They are usually sharp for the linear programming bound and also spherical designs (Delsarte-Goethals-Seidel, Levenshtein).
The new results in sphere packing

Theorem (Viazovska)

*The $E_8$ lattice packing is the densest sphere packing in $\mathbb{R}^8$.***
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The proof is fairly direct, using just two main ingredients:

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The proof is fairly direct, using just two main ingredients:

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2. the theory of modular forms
Linear programming bounds for sphere packing

Let the Fourier transform of a function $f$ be defined by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x, t \rangle} \, dx.$$
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Here is a plot of $\log(\text{density})$ vs. dimension.

Look at slopes (asymptotically) as well as where these curves meet.

Conjecture (Cohn-Elkies)
There exist "magic" functions $f_{E_8}$ and $f_{\Lambda_{24}}$ whose corresponding upper bounds match the densities of $E_8$ and $\Lambda_{24}$. 
LP bounds with dimension

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Desired functions

Let $\Lambda$ be $E_8$ or the Leech lattice, and $r_0, r_1, \ldots$ its nonzero vector lengths (square roots of the even natural numbers, except Leech skips 2). To have a tight upper bound that matches $\Lambda$, we need the function $f$ to look like this:
Desired functions

While $\hat{f}$ must look like this:
In [Cohn-Kumar] we used a polynomial of degree 803 and 3000 digits of precision to find $f$ and $\hat{f}$ which looked like this with 200 forced double roots, and $r$ very close to 2.
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We were stuck for more than a decade.
Viazovska’s breakthrough

In March 2016 Maryna Viazovska posted a preprint to the arxiv, solving the sphere packing problem in 8 dimensions.

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 Shortly afterward Cohn-Kumar-Miller-Radchenko-Viazovska were able to adapt her ideas to find the magic function $f_{24}$, solving the 24-dimensional sphere packing problem.
Modular group

A modular form is a function $\phi : \mathcal{H} \to \mathbb{C}$ with a lot of symmetries.
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Fundamental domain

The picture shows Dedekind’s famous tesselation of the upper half plane. The union of a black and a white region makes a fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$.
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The quotient $\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ can be identified with the Riemann sphere $\mathbb{C}P^1$ minus a point.

The preimages of this point are $\infty$ and the rational numbers; these are the cusps.
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We say $\Gamma$ is a congruence subgroup if it contains all the elements of $SL_2(\mathbb{Z})$ congruent to the identity modulo $N$, for some natural number $N$. Again the quotient is a complex algebraic curve; we can compactify it by adding finitely many cusps.
The first condition for a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ to be a modular form for $\Gamma$ of weight $k$ is

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z)$$

for all matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

(That is, these are the infinitely many “symmetries” of $f$.)
Modular forms

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The second condition is a growth condition as we approach a cusp, which we won’t describe in detail here.
Examples

How do we find actual examples of modular forms?
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One way is to take simple examples of a “well-behaved” holomorphic function and symmetrize (recalling that $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$):

$$G_k(z) = \sum_{(a,b) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(az + b)^k}.$$
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For even $k \geq 4$, the sum converges absolutely and we get a non-zero modular form of weight $k$. These are called Eisenstein series.
Another way is by taking **theta functions** of integral lattices.

If $\Lambda$ is a lattice whose inner products $\langle x, y \rangle$ are all integers, then

$$
\Theta_{\Lambda}(z) = \sum_{v \in \Lambda} \exp(\pi i z |v|^2)
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is a modular form for a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. 

**Fact** The space of modular forms of a given weight for a given congruence subgroup is finite dimensional.
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\textbf{Fact}

*The space of modular forms of a given weight for a given congruence subgroup is finite dimensional.*
Sketch of proof

The magic function is constructed using Laplace transforms of suitable modular forms: \( f_8 = f_{8, +} + f_{8, -} \), where

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f_{8, \epsilon} = \sin^2\left(\frac{\pi r^2}{2}\right) \int_0^\infty \phi_\epsilon(it) e^{-\pi r^2 t} \, dt
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for \( \epsilon \in \{\pm 1\} \), where \( \phi_\epsilon \) is a modular form chosen for a suitable congruence subgroup and suitable weight such that
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- The integral has a simple pole at $r^2 = 2$. (Note that $\sin^2(\pi r^2/2)$ has double zeros at $r^2 = 2, 4, 6, 8, \ldots$, so this makes $f$ have the correct “shape”.)
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- The symmetry properties of $\phi_\epsilon$ imply that $f_{8,\epsilon} = \hat{f}_{8,\epsilon} = \epsilon f_{8,\epsilon}$, i.e., it is an eigenfunction for the Fourier transform.

One still has to show there are no extra sign changes, which can be accomplished by a computer check of appropriate properties of $\phi$. 
Future work

There is a broad generalization of sphere packings - the problem of potential energy minimization. Cohn and I conjectured that $E_8$ and the Leech lattices and the hexagonal lattice $A_2$ are optimal for a wide range of potentials (and proved analogous statements for many optimal spherical codes).

Critical ingredient is LP bounds for energy (due to Yudin for sphere, Cohn-K for Euclidean space).
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We also have a project to analyze whether these modular-form techniques will lead to asymptotic improvements in upper bounds for sphere packing density.
References


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Thank you!